

ON THE PERIODIC SOLUTIONS OF A GENERALIZED SMOOTH OR NON-SMOOTH PERTURBED PLANAR DOUBLE PENDULUM

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ABSTRACT. We provide sufficient conditions for the existence of periodic solutions with small amplitude of the non-linear planar double pendulum perturbed by smooth or non-smooth functions.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We consider a system of two point masses m_1 and m_2 moving in a fixed plane, in which the distance between a point P (called pivot) and m_1 and the distance between m_1 and m_2 are fixed, and equal to l_1 and l_2 respectively. We assume the masses do not interact. We allow gravity to act on the masses m_1 and m_2 . This system is called the planar *double pendulum*.

The position of the double pendulum is determined by the two angles ϕ_1 and ϕ_2 shown in Figure 1. The corresponding Lagrange equations of motion are

$$(1) \quad \begin{aligned} & (m_1 + m_2)l_1\ddot{\phi}_1 + m_2l_2\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + (m_1 + m_2)g \sin(\phi_1) \\ & + m_2l_2\dot{\phi}_2^2 \sin(\phi_1 - \phi_2) = 0, \\ & m_2l_1\ddot{\phi}_1 \cos(\phi_1 - \phi_2) + m_2l_2\ddot{\phi}_2 + m_2g \sin(\phi_2) + m_2l_1\dot{\phi}_1^2 \sin(\phi_1 - \phi_2) = 0, \end{aligned}$$

where g is the acceleration of the gravity. For more details on these equations of motion see [5]. Here the dot denotes derivative with respect to the time T .

The authors in [8] have studied, in the vicinity of the equilibrium $\phi_1 = \phi_2 = 0$, the persistence of periodic solutions of system (1) perturbed smoothly in the particular case when $m_1 = m_2$ and $l_1 = l_2$. Now m_1, m_2, l_1 and l_2 can take arbitrary positive values and we shall study the periodic orbits of system (1) which persist under smooth and non-smooth perturbations.

Denote the expressions for $\ddot{\phi}_1$ and $\ddot{\phi}_2$ in (1) respectively by $H_1(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)$ and $H_2(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)$. In this paper we shall consider the perturbed problem

$$(2) \quad \begin{aligned} \ddot{\phi}_1 = & H_1(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \varepsilon \left(\hat{F}_1(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \hat{F}_2(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \operatorname{sgn}(\phi_1) \right) \\ & + \varepsilon^2 \left(\hat{R}_1(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \hat{R}_2(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \operatorname{sgn}(\phi_1) \right) + \mathcal{O}(\varepsilon^3), \\ \ddot{\phi}_2 = & H_2(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \varepsilon \left(\hat{F}_3(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \hat{F}_4(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \operatorname{sgn}(\phi_2) \right) \\ & + \varepsilon^2 \left(\hat{R}_3(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \hat{R}_4(t, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \operatorname{sgn}(\phi_2) \right) + \mathcal{O}(\varepsilon^3). \end{aligned}$$

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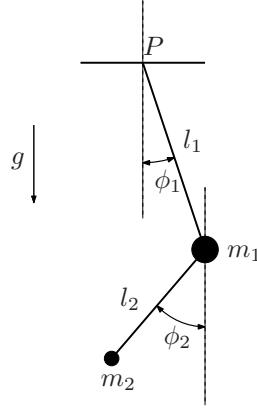


FIGURE 1. The planar double pendulum.

The function $\text{sgn}(z)$ denotes the sign function, i.e.

$$\text{sgn}(z) = \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

Here the smooth functions \hat{F}_i and \hat{R}_i for $i = 1, 2, 3, 4$ define the perturbation. These functions are respectively T_{F_i} -periodic and T_{R_i} -periodic in t and respectively in resonance $p_{F_i}:q_{F_i}$ and $p_{R_i}:q_{R_i}$ with some of the periodic solutions of the linearized unperturbed double pendulum, being p and q relatively prime positive integers for $p = p_{F_i}$, p_{R_i} , $q = q_{F_i}$, q_{R_i} and $i = 1, 2, 3, 4$. We also assume that $F_i(t, 0, 0, 0, 0) = 0$ for $i = 1, 2, 3, 4$.

Remark 1. For simplicity, we can assume that the functions \hat{F}_i and \hat{R}_i for $i = 1, 2, 3, 4$ are T -periodic with $T = pT_j$ for some integer p where T_j for $j = 1, 2$ are the periods of the solutions of the linearized unperturbed double pendulum. Indeed, if we take p the least common multiple among p_{F_i} and p_{R_i} for $i = 1, 2, 3, 4$, then there exists integers n_{F_i} and n_{R_i} such that $p = n_{F_i}p_{F_i} = n_{R_i}p_{R_i}$ for $i = 1, 2, 3, 4$. Hence

$$pT_j = n_{F_i}q_{F_i} \frac{p_{F_i}}{q_{F_i}} T_j = n_{R_i}q_{R_i} \frac{p_{R_i}}{q_{R_i}} T_j.$$

For $i = 1, 2, 3, 4$, and $j = 1, 2$.

Note that the functions \hat{F}_i and \hat{R}_i for $i = 1, 2, 3, 4$, can be taken in a certain way arbitrary, i.e., only assuming some hypotheses. It makes us able to provide, in a physical context, real meaning for these functions. In our case, since we are working with discontinuity in the variables θ_1 and θ_2 , the functions $\hat{F}_1, \hat{F}_2, \hat{R}_1$ and \hat{R}_2 could model the escapement for the particle m_1 , and the functions $\hat{F}_3, \hat{F}_4, \hat{R}_3$ and \hat{R}_4 could model the escapement for the particle m_2 . If we work with discontinuity in the variables θ'_1 and θ'_2 , instead with discontinuity in the variables θ_1 and θ_2 , the respective functions could model the Coulomb Friction. We also can work composing these two phenomena. For more details on physical systems with discontinuous models see, for instance, [1] and [2].

Now, we follow the steps:

(i) proceed with the change of variable $\phi_1 = \varepsilon\theta_1$ and $\phi_2 = \varepsilon\theta_2$;

$$\begin{aligned}\ddot{\theta}_1 &= \frac{1}{\varepsilon} H_1(\varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \hat{F}_1(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \hat{F}_2(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2)\text{sgn}(\theta_1) \\ &\quad + \varepsilon \left(\hat{R}_1(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \hat{R}_2(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2)\text{sgn}(\theta_1) \right) + \mathcal{O}(\varepsilon^3), \\ \ddot{\theta}_2 &= \frac{1}{\varepsilon} H_2(\varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + F_3(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + F_4(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2)\text{sgn}(\theta_2) \\ &\quad + \varepsilon \left(\hat{R}_1(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \hat{R}_2(t, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2)\text{sgn}(\theta_2) \right) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

(ii) expand in Taylor series, for $\varepsilon = 0$, the expressions of $\ddot{\theta}_1$ and $\ddot{\theta}_2$;

(iii) Take a new time t given by the rescaling $t = \alpha\tau$, with $\alpha = \sqrt{l_1 m_1 / (g m_2)}$;

(iv) and finally, denote

$$a = \frac{m_1 + m_2}{m_2} > 1 \quad \text{and} \quad b = \frac{l_1(m_1 + m_2)}{l_2 m_2} > 0.$$

Hence, we obtain the following equations of motion for the double pendulum

(3)

$$\begin{aligned}\theta_1'' &= -a\theta_1 + \theta_2 + \varepsilon(K_1(\tau) + F_1(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2) + (K_2(\tau) + F_2(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2))\text{sgn}(\theta_1)) \\ &\quad + \varepsilon^2 R_1(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2, \varepsilon), \\ \theta_2'' &= b\theta_1 - b\theta_2 + \varepsilon(K_3(\tau) + F_3(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2) + (K_4(\tau) + F_4(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2))\text{sgn}(\theta_2)) \\ &\quad + \varepsilon^2 R_2(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2, \varepsilon),\end{aligned}$$

where now the prime denotes derivative with respect to the new time t . Here the functions F_i , for $i = 1, 2, 3, 4$, are linear in the spatial variables, and are given by

$$F_i(\tau, \theta_1, \theta'_1, \theta_2, \theta'_2) = d_i^1(\tau)\theta_1 + d_i^2(\tau)\theta'_1 + d_i^3(\tau)^3\theta_2 + d_i(\tau)^4\theta'_2.$$

with

$$\begin{aligned}d_i^1(\tau) &= \alpha^2 \frac{\partial \hat{F}_i}{\partial \phi_1}(\alpha\tau, \vec{0}), \quad d_i^2(\tau) = \alpha \frac{\partial \hat{F}_i}{\partial \dot{\phi}_1}(\alpha\tau, \vec{0}), \\ d_i^3(\tau) &= \alpha^2 \frac{\partial \hat{F}_i}{\partial \phi_2}(\alpha\tau, \vec{0}), \quad d_i^4(\tau) = \alpha \frac{\partial \hat{F}_i}{\partial \dot{\phi}_2}(\alpha\tau, \vec{0}),\end{aligned}$$

and $K_i(\tau) = \alpha^2 \hat{R}_i(\alpha\tau, 0, 0, 0)$. Observe that, for $i, j = 1, 2, 3, 4$, $d_i^j(\tau)$ and $K_i(\tau)$ are T/α -periodic functions.

The objective of this paper is to provide a system of equations whose simple zeros provide periodic solutions (see Figure 2) of the perturbed planar double pendulum (2).

In order to present our results we need some preliminary definitions and notations.

The unperturbed system (3) has a unique singular point, the origin with eigenvalues $\pm\omega_1 i, \pm\omega_2 i$, where

$$\omega_1 = \frac{\sqrt{a+b-\sqrt{\Delta}}}{\sqrt{2}}, \quad \omega_2 = \frac{\sqrt{a+b+\sqrt{\Delta}}}{\sqrt{2}},$$

with $\Delta = (a-b)^2 + 4b > 0$. Consequently this system in the phase space $(\theta_1, \theta'_1, \theta_2, \theta'_2)$ has two planes filled with periodic solutions except the origin. The periods of such periodic orbits are

$$T_1 = \frac{2\pi}{\omega_1} \quad \text{or} \quad T_2 = \frac{2\pi}{\omega_2}.$$

These periodic orbits live into the planes associated to the eigenvectors with eigenvalues $\pm\omega_1 i$ or $\pm\omega_2 i$, respectively. We shall study which of these periodic solutions persist for the perturbed

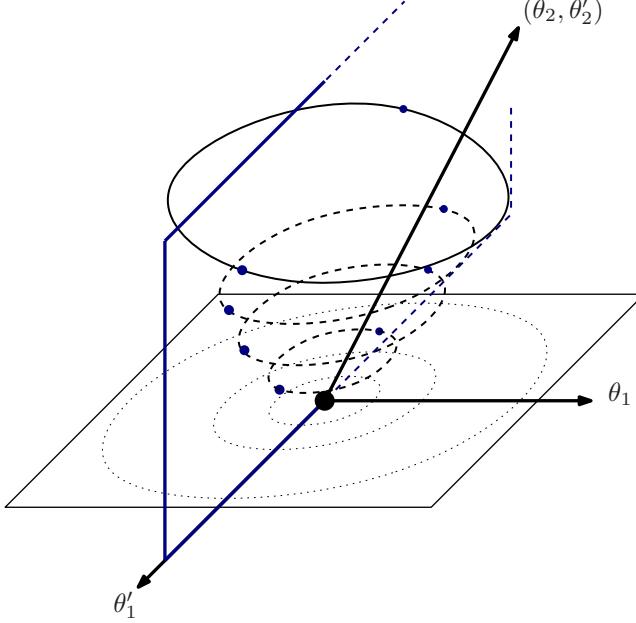


FIGURE 2. Periodic solution of the perturbed system (2) converging to the origin, when $\varepsilon \rightarrow 0$.

system (2) when the parameter ε is sufficiently small and the functions of perturbation \hat{F}_i and \hat{R}_i for $i = 1, 2, 3, 4$ have period either $p\alpha T_1$, or $p\alpha T_2$, with p positive integer.

Remark 2. We say that the *Crossing Hypothesis* is satisfied if there exists a compact set $D \subset \mathbb{R}^4$ such that every orbit starting in D reaches the set of discontinuity only at its crossing regions (see Appendix A).

Let $X_{X_0, Y_0}(\tau)$ be the periodic function

$$X_{X_0, Y_0}(\tau) = Y_0 \cos(\omega_1 \tau) + X_0 \sin(\omega_1 \tau),$$

then we define the non-smooth function $\mathcal{F}_1(X_0, Y_0)$ by

$$(4) \quad \begin{aligned} & \int_0^{pT_1} \sin(\omega_1 \tau) \left(2b(\bar{F}_1 + K_1(\tau)) + (\bar{F}_3 + K_3(\tau)) (a - b + \sqrt{\Delta}) \right) d\tau \\ & + \int_0^{pT_1} \sin(\omega_1 \tau) \left(2b(\bar{F}_2 + K_2(\tau)) + (\bar{F}_4 + K_4(\tau)) (a - b + \sqrt{\Delta}) \right) \operatorname{sgn}(X_{X_0, Y_0}(\tau)) d\tau, \end{aligned}$$

and the non-smooth function $\mathcal{F}_2(X_0, Y_0)$ by

$$(5) \quad \begin{aligned} & \int_0^{pT_1} \cos(\omega_1 \tau) \left(2b(\bar{F}_1 + K_1(\tau)) + (\bar{F}_3 + K_3(\tau)) (a - b + \sqrt{\Delta}) \right) d\tau \\ & + \int_0^{pT_1} \cos(\omega_1 \tau) \left(2b(\bar{F}_2 + K_2(\tau)) + (\bar{F}_4 + K_4(\tau)) (a - b + \sqrt{\Delta}) \right) \operatorname{sgn}(X_{X_0, Y_0}(\tau)) d\tau. \end{aligned}$$

where

$$(6) \quad \bar{F}_i = F_i(\tau, A_1, B_1, C_1, D_1)$$

for $i = 1, 2, 3, 4$ with

$$\begin{aligned} A_1 &= \frac{(-a + b + \sqrt{\Delta})}{2b\omega_1} (X_0 \cos(\omega_1 \tau) + Y_0 \sin(\omega_1 \tau)), \\ B_1 &= \frac{(-a + b + \sqrt{\Delta})}{2b} (Y_0 \cos(\omega_1 \tau) - X_0 \sin(\omega_1 \tau)), \\ C_1 &= \frac{1}{\omega_1} (X_0 \cos(\omega_1 \tau) + Y_0 \sin(\omega_1 \tau)), \\ D_1 &= Y_0 \cos(\omega_1 \tau) - X_0 \sin(\omega_1 \tau). \end{aligned}$$

A zero (X_0^*, Y_0^*) of the system of the non-smooth functions

$$(7) \quad \mathcal{F}_1(X_0, Y_0) = 0, \quad \mathcal{F}_2(X_0, Y_0) = 0,$$

such that

$$\det \left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(X_0, Y_0)} \Big|_{(X_0, Y_0) = (X_0^*, Y_0^*)} \right) \neq 0,$$

is called a *simple zero* of system (7).

Our main result on the periodic solutions of the non-smooth perturbed double pendulum (2) which bifurcate from the periodic solutions of the unperturbed double pendulum (1) with period T_1 traveled p times is the following.

Theorem 1. *Assume that the functions \hat{F}_i and \hat{R}_i of the non-smooth perturbed double pendulum (2) are periodic in t of period $p\alpha T_1$ with p positive integer. Also assume that the Crossing Hypothesis (see Remark 2) is satisfied. Then for $|\varepsilon| > 0$ sufficiently small and for every simple zero $(X_0^*, Y_0^*) \neq (0, 0)$ of the non-smooth system (7) such that the orbits pass by D , the non-smooth perturbed double pendulum (2) has a $p\alpha T_1$ -periodic solution $(\phi_1(t, \varepsilon), \phi_2(t, \varepsilon)) \rightarrow (0, 0)$ when $\varepsilon \rightarrow 0$.*

Theorem 1 is proved in section 2. Its proof is based in the averaging theory for computing periodic solutions, see the Appendix B.

Note that the periodic solution given in Theorem 1 is a periodic solution bifurcating at $\varepsilon = 0$ from the equilibrium of system (2) localized at the origin of coordinates. For $|\varepsilon| > 0$ sufficiently small this orbits is close to the plane defined by the eigenvectors of the eigenvalues $\pm i\omega_1$.

We provide an application of Theorem 1 in the following corollary, which will be proved in section 3.

Corollary 2. *Suppose that $F_1 = y/\alpha^2 + f_1$, $F_3 = w/\alpha^2 + f_3$, and f_1, F_2, f_3 , and F_4 has no linear term. Also suppose that $R_2 = 1/\alpha^2 + r_2$, $R_4 = 1/\alpha^2 + r_4$ and R_1, r_2, R_3 , and r_4 has no constant term. Moreover, assume that all functions are $p\alpha T_1/q$ -periodic. Then the differential system (2) for $|\varepsilon| > 0$ sufficiently small has one $p\alpha T_1$ -periodic solution $(\phi_1(t, \varepsilon), \phi_2(t, \varepsilon)) \rightarrow (0, 0)$ when $\varepsilon \rightarrow 0$.*

Now let $Z^{Z_0, W_0}(\tau)$ be the periodic function

$$Z^{Z_0, W_0}(\tau) = W_0 \cos(\omega_2 \tau) + Z_0 \sin(\omega_2 \tau),$$

then we define the non-smooth function $\mathcal{F}^1(Z_0, W_0)$ by

$$(8) \quad \begin{aligned} & \int_0^{pT_2} \sin(\omega_2 \tau) \left(-2b(\bar{F}_1 + K_1(\tau)) + (\bar{F}_3 + K_3(\tau))(-a + b + \sqrt{\Delta}) \right) d\tau \\ & + \int_0^{pT_2} \sin(\omega_2 \tau) \left(2b(\bar{F}_2 + K_2(\tau)) + (\bar{F}_4 + K_4(\tau))(-a + b + \sqrt{\Delta}) \right) \operatorname{sgn}(Z^{Z_0, W_0}(\tau)) d\tau, \end{aligned}$$

and the non-smooth function $\mathcal{F}^2(Z_0, W_0)$ by

$$(9) \quad \begin{aligned} & \int_0^{pT_2} \cos(\omega_2 \tau) \left(-2b(\bar{F}_1 + K_1(\tau)) + (\bar{F}_3 + K_3(\tau))(a - b + \sqrt{D}) \right) d\tau \\ & + \int_0^{pT_2} \cos(\omega_2 \tau) \left(2b(\bar{F}_2 + K_2(\tau)) + (\bar{F}_4 + K_4(\tau))(a - b + \sqrt{D}) \right) \operatorname{sgn}(Z^{Z_0, W_0}(\tau)) d\tau. \end{aligned}$$

where

$$\bar{F}_i = F_i(t, A_2, B_2, C_2, D_2)$$

for $i = 1, 2, 3, 4$ with

$$\begin{aligned} A_2 &= -\frac{(a - b + \sqrt{\Delta})}{2b\omega_2} (Z_0 \cos(\omega_2 t) + W_0 \sin(\omega_2 t)), \\ B_2 &= -\frac{(a - b + \sqrt{\Delta})}{2b} (W_0 \cos(\omega_2 t) - Z_0 \sin(\omega_2 t)), \\ C_2 &= \frac{1}{\omega_2} (Z_0 \cos(\omega_2 t) + W_0 \sin(\omega_2 t)), \\ D_2 &= W_0 \cos(\omega_2 t) - Z_0 \sin(\omega_2 t). \end{aligned}$$

Consider the non-linear and non-smooth system

$$(10) \quad \mathcal{F}^1(Z_0, W_0) = 0, \quad \mathcal{F}^2(Z_0, W_0) = 0.$$

Our main result on the periodic solutions of the non-smooth perturbed double pendulum (2) which bifurcate from the periodic solutions of the unperturbed double pendulum (3) with period T_2 traveled p times is the following.

Theorem 3. *Assume that the functions \hat{F}_i and \hat{R}_i of the non-smooth perturbed double pendulum (2) are periodic in t of period $p\alpha T_2$ with p positive integer. Also assume that the Crossing Hypothesis (see Remark 2) is satisfied. Then for $\varepsilon > 0$ sufficiently small and for every simple zero $(Z_0^*, W_0^*) \neq (0, 0)$ of the non-smooth system (10) such that the orbits pass by D , the non-smooth perturbed double pendulum (2) has a $p\alpha T_2$ -periodic solution $(\phi_1(t, \varepsilon), \phi_2(t, \varepsilon)) \rightarrow (0, 0)$ when $\varepsilon \rightarrow 0$.*

Theorem 3 is also proved in section 2.

Again the periodic solution given in Theorem 3 is a periodic solution bifurcating at $\varepsilon = 0$ from the equilibrium of system (2) localized at the origin of coordinates. For $|\varepsilon| > 0$ sufficiently small this orbits is close to the plane defined by the eigenvectors of the eigenvalues $\pm i\omega_2$.

We provide an application of Theorem 3 in the following corollary, which will be proved in section 3.

Corollary 4. *Suppose that $F_1 = y/\alpha^2 + f_1$, $F_3 = w/\alpha^2 + f_3$, and f_1, F_2, f_3 , and F_4 has no linear term. Also suppose that $R_2 = 1/\alpha^2 + r_2$, $R_4 = 1/\alpha^2 + r_4$ and R_1, r_2, R_3 , and r_4 has no constant term. Moreover, assume that all functions are $p\alpha T_2/q$ -periodic. Then the differential system*

(2) for $|\varepsilon| > 0$ sufficiently small has one $p\alpha T_2$ -periodic solution $(\phi_1(t, \varepsilon), \phi_2(t, \varepsilon)) \rightarrow (0, 0)$ when $\varepsilon \rightarrow 0$.

2. PROOFS OF THEOREMS 1 AND 3

Introducing the variables $(x, y, z, w) = (\theta_1, \theta'_1, \theta_2, \theta'_2)$ we write the differential system of the non-smooth perturbed double pendulum (3) as a first-order differential system defined in \mathbb{R}^4 . Thus we have the differential system

$$(11) \quad \begin{aligned} x' &= y, \\ y' &= -ax + z + \varepsilon(K_1(\tau) + F_1(\tau, x, y, z, w) + (K_2(\tau) + F_2(\tau, x, y, z, w)) \operatorname{sgn}(x)) \\ &\quad + \varepsilon^2 R_1(\tau, x, y, z, w, \varepsilon), \\ z' &= w, \\ w' &= bx - bz + \varepsilon(K_3(\tau) + F_3(\tau, x, y, z, w) + (K_4(\tau) + F_4(\tau, x, y, z, w)) \operatorname{sgn}(z)) \\ &\quad + \varepsilon^2 R_2(\tau, x, y, z, w, \varepsilon). \end{aligned}$$

System (11) with $\varepsilon = 0$ is equivalent to the unperturbed double pendulum system (3), called in what follows simply by the *unperturbed system*. Otherwise we have the *perturbed system*.

Instead of working with the discontinuous differential system (11) we shall work with the smooth differential system

$$(12) \quad \begin{aligned} x' &= y, \\ y' &= -ax + z + \varepsilon(K_1(\tau) + F_1(\tau, x, y, z, w) + (K_2(\tau) + F_2(\tau, x, y, z, w)) s_\delta(y)) \\ &\quad + \varepsilon R_1(\tau, x, y, z, w, \varepsilon), \\ z' &= w, \\ w' &= bx - bz + \varepsilon(K_3(\tau) + F_3(\tau, x, y, z, w) + (K_4(\tau) + F_4(\tau, x, y, z, w)) s_\delta(w)) \\ &\quad + \varepsilon R_2(\tau, x, y, z, w, \varepsilon). \end{aligned}$$

where $s_\delta(x)$ is the smooth function defined in Figure 5, such that

$$\lim_{\delta \rightarrow 0} s_\delta(x) = \operatorname{sgn}(x).$$

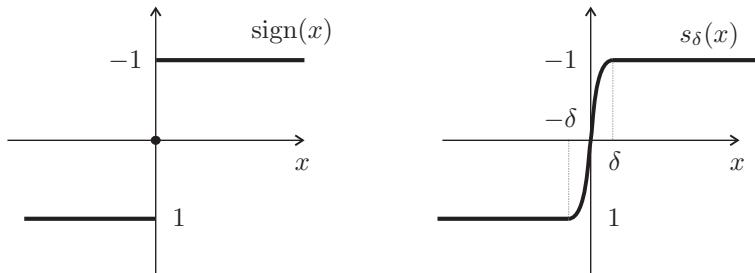


FIGURE 3. The functions $\operatorname{sgn}(x)$ and $s_\delta(x)$.

We shall write system (12) in such a way that the linear part at the origin of the unperturbed system will be in its real normal Jordan form. Then, doing the change of variables $(\tau, x, y, z, w) \rightarrow (\tau, X, Y, Z, W)$ given by

$$(13) \quad \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{pmatrix} \frac{b\omega_1}{\sqrt{\Delta}} & 0 & \frac{\omega_1(a-b+\sqrt{\Delta})}{2\sqrt{\Delta}} & 0 \\ 0 & \frac{b}{\sqrt{\Delta}} & 0 & \frac{a-b+\sqrt{\Delta}}{2\sqrt{\Delta}} \\ -\frac{b\omega_2}{\sqrt{\Delta}} & 0 & \frac{\omega_2(-a+b+\sqrt{\Delta})}{2\sqrt{\Delta}} & 0 \\ 0 & -\frac{b}{\sqrt{\Delta}} & 0 & \frac{-a+b+\sqrt{\Delta}}{2\sqrt{\Delta}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

the differential system (12) becomes

$$(14) \quad \begin{aligned} X' &= \omega_1 Y, \\ Y' &= -\omega_1 X + \varepsilon \frac{1}{2\sqrt{\Delta}} \left(2b \left(K_1(\tau) + \tilde{F}_1 + (K_2(\tau) + \tilde{F}_2) s_\delta(\mathcal{A}) \right) \right) \\ &\quad + \varepsilon \frac{1}{2\sqrt{\Delta}} \left((a-b+\sqrt{\Delta}) \left(K_3(\tau) + \tilde{F}_3 + (K_4(\tau) + \tilde{F}_4) s_\delta(\mathcal{C}) \right) \right) + \varepsilon^2 \tilde{R}_1, \\ Z' &= \omega_2 W, \\ W' &= -\omega_2 Z + \varepsilon \frac{1}{2\sqrt{\Delta}} \left(-2b \left(K_1(\tau) + \tilde{F}_1 + (K_2(\tau) + \tilde{F}_2) s_\delta(\mathcal{A}) \right) \right) \\ &\quad + \varepsilon \frac{1}{2\sqrt{\Delta}} \left((-a+b+\sqrt{\Delta}) \left(K_3(\tau) + \tilde{F}_3 + (K_4(\tau) + \tilde{F}_4) s_\delta(\mathcal{C}) \right) \right) + \varepsilon^2 \tilde{R}_2, \end{aligned}$$

where $\tilde{F}_i(t, X, Y, Z, W) = F_i(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ for $i = 1, 2, 3, 4$ with

$$\begin{aligned} \mathcal{A} &= \frac{(-a+b+\sqrt{\Delta})}{2b\omega_1} X - \frac{(a-b+\sqrt{\Delta})}{2b\omega_2} Z, \\ \mathcal{B} &= \frac{(-a+b+\sqrt{\Delta})}{2b} Y - \frac{(a-b+\sqrt{\Delta})}{2b} W, \\ \mathcal{C} &= \frac{1}{\omega_1} X + \frac{1}{\omega_2} Z, \end{aligned}$$

$$\mathcal{D} = Y + W,$$

and $\tilde{R}_i = \tilde{R}_i(X, Y, Z, W, \varepsilon)$. Note that the linear part of the differential system (14) at the origin is in its real normal Jordan form.

Lemma 5. *The periodic solutions of the differential system (14) with $\varepsilon = 0$ are*

$$(15) \quad \begin{aligned} X_{X_0, Y_0}(\tau) &= X_0 \cos(\omega_1 \tau) + Y_0 \sin(\omega_1 \tau), \\ Y_{X_0, Y_0}(\tau) &= Y_0 \cos(\omega_1 \tau) - X_0 \sin(\omega_1 \tau), \\ Z_{X_0, Y_0}(\tau) &= 0, \\ W_{X_0, Y_0}(\tau) &= 0, \end{aligned}$$

of period T_1 , and

$$(16) \quad \begin{aligned} X^{Z_0, W_0}(\tau) &= 0, \\ Y^{Z_0, W_0}(\tau) &= 0, \\ Z^{Z_0, W_0}(\tau) &= Z_0 \cos(\omega_2 \tau) + W_0 \sin(\omega_2 \tau), \\ W^{Z_0, W_0}(\tau) &= W_0 \cos(\omega_2 \tau) - Z_0 \sin(\omega_2 \tau), \end{aligned}$$

of period T_2 .

Proof. Since system (14) with $\varepsilon = 0$ is a linear differential system, the proof follows easily. \square

Proof of Theorem 1. Assume that the functions \hat{F}_i and \hat{R}_i of the non-smooth perturbed double pendulum with equations of motion (3) are periodic in t of period $p\alpha T_1$ with p positive integer. Thus K_i and F_i are pT_1 -periodic functions, i.e., the differential system (3) and the periodic solutions (15) have the same period pT_1 .

It is well known that a Poincaré map defined in a smooth differential system is smooth. So the Poincaré maps associated to the periodic orbits of the differential system (12) are smooth.

The Poincaré maps, restricted at D , associated to the periodic solutions of the non-smooth differential system (11), which are perturbations of the periodic solutions (15) are also smooth. Since the orbits starting in D reaches the discontinuity set only at the crossing region (see Appendix A), such Poincaré maps are compositions of smooth Poincaré maps. In a similar way it follows that the Poincaré maps, restricted at D , associated to the periodic solutions of the non-smooth differential system (11), which are perturbations of the periodic solutions (16) are also smooth.

We can use Theorem 6 (see the Appendix B) for computing some periodic solutions of the smooth systems. The periodic solutions are zeros of the displacement function, which is the Poincaré map associated to periodic solutions minus the identity. In fact, the non-linear function (25) whose zeros can provide periodic solutions, is the first term of order ε of the displacement function. See for more details the proof of Theorem 6 in [3].

Since the Poincaré maps associated to periodic solutions of system (11), coming from the perturbed periodic solutions (15) or (16), are smooth and these Poincaré maps are the limit of the Poincaré maps associated to the smooth system (12), for which we can use Theorem 6, it follows that we also can use Theorem 6 for computing some of the periodic solutions of the non-smooth system (11). In other words, we can apply Theorem 6 to the smooth systems (12) and then pass to the limit, when $\delta \rightarrow 0$, the function (25) for obtaining a function whose zeros can give periodic solutions of the non-smooth system (11).

We shall apply Theorem 6 of the Appendix B to differential system (14). We note that system (14) can be written as system (22) taking

$$\mathbf{x} = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}, \quad t = \tau, \quad G_0(t, \mathbf{x}) = \begin{pmatrix} \omega_1 Y, \\ -\omega_1 X, \\ \omega_2 W, \\ -\omega_2 Z \end{pmatrix},$$

$$G_1(t, \mathbf{x}) = \begin{pmatrix} 0 \\ \frac{b}{\sqrt{\Delta}} \left(K_1(\tau) + \tilde{F}_1 + (K_2(\tau) + \tilde{F}_2) s_\delta(\mathcal{A}) \right) \\ + \frac{a-b+\sqrt{\Delta}}{2\sqrt{\Delta}} \left(K_3(\tau) + \tilde{F}_3 + (K_4(\tau) + \tilde{F}_4) s_\delta(\mathcal{C}) \right) \\ 0 \\ -\frac{b}{\sqrt{\Delta}} \left(K_1(\tau) + \tilde{F}_1 + (K_2(\tau) + \tilde{F}_2) s_\delta(\mathcal{A}) \right) \\ + \frac{-a+b+\sqrt{\Delta}}{2\sqrt{\Delta}} \left(K_3(\tau) + \tilde{F}_3 + (K_4(\tau) + \tilde{F}_4) s_\delta(\mathcal{C}) \right) \end{pmatrix}$$

and $G_2(t, \mathbf{x}, \varepsilon) = \begin{pmatrix} 0 \\ R_1 \\ 0 \\ R_2 \end{pmatrix}.$

We shall study which periodic solutions (15) of the unperturbed system (14) with $\varepsilon = 0$ can be continued to periodic solutions of the perturbed system (14) for $\varepsilon \neq 0$ sufficiently small.

We shall describe the different elements which appear in the statement of Theorem 6 in the particular case of the differential system (14). Thus we have that $\Omega = \mathbb{R}^4$, $k = 2$ and $n = 4$. Let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. We take the open and bounded subset V of the plane $Z = W = 0$ as

$$V = \{(X_0, Y_0, 0, 0) \in \mathbb{R}^4 : r_1 < \sqrt{X_0^2 + Y_0^2} < r_2\}.$$

As usual $\text{Cl}(V)$ denotes the closure of V . If $\alpha = (X_0, Y_0)$, then we can identify V with the set

$$\{\alpha \in \mathbb{R}^2 : r_1 < \|\alpha\| < r_2\},$$

here $\|\cdot\|$ denotes the Euclidean norm of \mathbb{R}^2 . The function $\beta : \text{Cl}(V) \rightarrow \mathbb{R}^2$ is $\beta(\alpha) = (0, 0)$. Therefore, in our case the set

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \text{Cl}(V)\} = \{(X_0, Y_0, 0, 0) \in \mathbb{R}^4 : r_1 \leq \sqrt{X_0^2 + Y_0^2} \leq r_2\}.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider the periodic solution $\mathbf{x}(\tau, \mathbf{z}_\alpha) = (X(\tau), Y(\tau), 0, 0)$ given by (15) with period pT_1 .

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\tau)$ of the linear differential system (14) with $\varepsilon = 0$ associated to the T -periodic solution $\mathbf{z}_\alpha = (X_0, Y_0, 0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of \mathbb{R}^4 , we get that $M(\tau) = M_{\mathbf{z}_\alpha}(\tau)$ is equal to

$$\begin{pmatrix} \cos(\omega_1 \tau) & \sin(\omega_1 \tau) & 0 & 0 \\ -\sin(\omega_1 \tau) & \cos(\omega_1 \tau) & 0 & 0 \\ 0 & 0 & \cos(\omega_2 \tau) & \sin(\omega_2 \tau) \\ 0 & 0 & -\sin(\omega_2 \tau) & \cos(\omega_2 \tau) \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\tau)$ does not depend on the particular periodic solution $\mathbf{x}(\tau, \mathbf{z}_\alpha)$. Since the matrix

$$M^{-1}(0) - M^{-1}(pT_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 \sin^2 \left(\frac{p\pi \omega_2}{\omega_1} \right) & \sin \left(\frac{2p\pi \omega_2}{\omega_1} \right) \\ 0 & 0 & -\sin \left(\frac{2p\pi \omega_2}{\omega_1} \right) & 2 \sin^2 \left(\frac{p\pi \omega_2}{\omega_1} \right) \end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 6 because the determinant

$$\begin{vmatrix} 2 \sin^2 \left(\frac{p\pi \omega_2}{\omega_1} \right) & \sin \left(\frac{2p\pi \omega_2}{\omega_1} \right) \\ -\sin \left(\frac{2p\pi \omega_2}{\omega_1} \right) & 2 \sin^2 \left(\frac{p\pi \omega_2}{\omega_1} \right) \end{vmatrix} = 4 \sin^2 \left(\frac{p\pi \omega_2}{\omega_1} \right) \neq 0.$$

So we can apply Theorem 6 to system (14).

Now $\xi : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is $\xi(X, Y, Z, W) = (X, Y)$. We calculate, when $\delta \rightarrow 0$, the function

$$\mathcal{G}(X_0, Y_0) = \mathcal{G}(\alpha) = \xi \left(\frac{1}{pT_1} \int_0^{pT_1} M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right),$$

and we obtain the function $\mathcal{G}_2(X_0, Y_0)$

$$(17) \quad -\frac{1}{2\sqrt{\Delta}pT_1} \int_0^{pT_1} \left[\sin(\omega_1 \tau) \left(2b \left(K_1(\tau) + \bar{F}_1 + (K_2(\tau) + \bar{F}_2) \operatorname{sgn} \left(\frac{a+b+\sqrt{\Delta}}{2b} X_{X_0, Y_0}(\tau) \right) \right) \right. \right. \\ \left. \left. + (a-b+\sqrt{\Delta}) (K_3(\tau) + \bar{F}_3 + (K_4(\tau) + \bar{F}_4) \operatorname{sgn}(X_{X_0, Y_0}(\tau))) \right) \right] d\tau,$$

and the function $\mathcal{G}_2(X_0, Y_0)$

$$(18) \quad \frac{1}{2\sqrt{\Delta}pT_1} \int_0^{pT_1} \left[\cos(\omega_1 \tau) \left(+2b \left(K_1(\tau) + \bar{F}_1 + (K_2(\tau) + \bar{F}_2) \operatorname{sgn} \left(\frac{a+b+\sqrt{\Delta}}{2b} X_{X_0, Y_0}(\tau) \right) \right) \right. \right. \\ \left. \left. + (a-b+\sqrt{\Delta}) (K_3(\tau) + \bar{F}_3 + (K_4(\tau) + \bar{F}_4) \operatorname{sgn}(X_{X_0, Y_0}(\tau))) \right) \right] d\tau,$$

where the functions of \bar{F}_i for $i = 1, 2, 3, 4$ are the ones given in (6). Note that $-a+b+\sqrt{\Delta} > 0$, then

$$\operatorname{sgn} \left(\frac{-a+b+\sqrt{\Delta}}{2b} X_{X_0, Y_0}(\tau) \right) = \operatorname{sgn}(X_{X_0, Y_0}(\tau)),$$

denoting by $K = 1/(2\sqrt{\Delta}pT_1)$, the function (17) $\mathcal{G}_1(X_0, Y_0)$ becomes

$$\begin{aligned} -K \int_0^{pT_1} \sin(\omega_1 \tau) (K_3(\tau) + \bar{F}_3) \left(a - b + \sqrt{\Delta} \right) + 2b(K_1(\tau) + \bar{F}_1) d\tau \\ -K \int_0^{pT_1} \sin(\omega_1 \tau) \left((K_4(\tau) + \bar{F}_4) \left(a - b + \sqrt{\Delta} \right) + 2b(K_2(\tau) + \bar{F}_2) \right) \operatorname{sgn}(X_{X_0, Y_0}(\tau)) d\tau, \end{aligned}$$

and the function (18) $\mathcal{G}_2(X_0, Y_0)$ becomes

$$\begin{aligned} & K \int_0^{pT_1} \cos(\omega_1 \tau) (K_3(\tau) + \bar{F}_3) \left(a - b + \sqrt{D} \right) + 2b(K_1(\tau) + \bar{F}_1) d\tau \\ & K \int_0^{pT_1} \cos(\omega_1 \tau) \left((K_4(\tau) + \bar{F}_4) \left(a - b + \sqrt{D} \right) + 2b(K_2(\tau) + \bar{F}_2) \right) \operatorname{sgn}(X_{X_0, Y_0}(\tau)) d\tau. \end{aligned}$$

Then, by Theorem 6 we have that for every simple zero $(X_0^*, Y_0^*) \in V$ of the system of non-linear and non-smooth functions

$$(19) \quad \mathcal{G}_1(X_0, Y_0) = 0, \quad \mathcal{G}_2(X_0, Y_0) = 0,$$

we have a periodic solution $(X, Y, Z, W)(t, \varepsilon)$ of system (14) such that

$$(X, Y, Z, W)(0, \varepsilon) \rightarrow (X_0^*, Y_0^*, 0, 0) \quad \text{as } \varepsilon \rightarrow 0.$$

Note that system (19) is equivalent to system (7), because both equations only differs in a non-zero multiplicative constant.

Going back through the change of coordinates (13) we get a periodic solution $(x, y, z, w)(\tau, \varepsilon)$ of system (15) such that

$$\begin{pmatrix} x(\tau, \varepsilon) \\ y(\tau, \varepsilon) \\ z(\tau, \varepsilon) \\ w(\tau, \varepsilon) \end{pmatrix} \rightarrow \begin{pmatrix} \frac{-a+b+\sqrt{\Delta}}{2b\omega_1} (X_0^* \cos(\omega_1 \tau) + Y_0^* \sin(\omega_1 \tau)) \\ \frac{-a+b+\sqrt{\Delta}}{2b} (Y_0^* \cos(\omega_1 \tau) - X_0^* \sin(\omega_1 \tau)) \\ \frac{1}{\omega_1} (X_0^* \cos(\omega_1 \tau) + Y_0^* \sin(\omega_1 \tau)) \\ Y_0^* \cos(\omega_1 \tau) - X_0^* \sin(\omega_1 \tau) \end{pmatrix}$$

as $\varepsilon \rightarrow 0$.

Consequently we obtain a periodic solution $(\theta_1, \theta_2)(\tau, \varepsilon)$ of system (2) such that

$$(\theta_1, \theta_2)(\tau, \varepsilon) \rightarrow \begin{pmatrix} \frac{-a+b+\sqrt{\Delta}}{2b\omega_1} (X_0^* \cos(\omega_1 \tau) + Y_0^* \sin(\omega_1 \tau)) \\ \frac{1}{\omega_1} (X_0^* \cos(\omega_1 \tau) + Y_0^* \sin(\omega_1 \tau)) \end{pmatrix}$$

as $\varepsilon \rightarrow 0$. Hence Theorem 1 is proved. \square

Proof of Theorem 3. This proof is completely analogous to the proof of Theorem 1. \square

3. PROOFS OF COROLLARIES

To obtain the expression of the functions given in (4) and (5) we have to study the changes of sign of the functions $X_{X_0, Y_0}(\tau)$ and $Z^{Z_0, W_0}(\tau)$ respectively for $t \in [0, pT_1]$ and $t \in [0, pT_2]$.

Note that $X_{X_0, Y_0}(t_n) = 0$ for

$$t_n = \frac{1}{\omega_1} \left(\pi n - \arctan \left(\frac{Y_0}{X_0} \right) \right).$$

If $X_0 Y_0 < 0$, then $t_n \in [0, pT_1]$ only for $n = 0, 1, \dots, p+1$, and if $X_0 Y_0 > 0$, then $t_n \in [0, pT_1]$ only for $n = 1, 2, \dots, p+2$. We know that for all $t \in [t_n, t_{n+1}]$ the function $Y_{X_0, Y_0}(\tau)$ has the same sign and different sign for any $t \in [t_{n-1}, t_n]$, thus the integral can be computed using the

partitions $\{0, t_n, pT_1; n = 0, 1, \dots, p+1\}$ and $\{0, t_n, pT_1; n = 1, 2, \dots, p+2\}$ as the limits of integration respectively for $X_0Y_0 < 0$ and $X_0Y_0 > 0$.

The study of changes of the sign of the function $Z^{Z_0, W_0}(\tau)$ for $t \in [0, pT_2]$ and $Z_0W_0 \neq 0$ is completely analogous.

Proof of Corollary 2. Firstly, we have to check the *Crossing Hypothesis* (see Remark 2) to the system (3).

Note that we have four different vector fields defined in four different regions (see Figure 4).

In the region $\mathcal{R}_1 = \{x > 0 \text{ and } z > 0\}$ we have

$$\mathcal{X}_1 = \begin{pmatrix} y \\ -ax + z + \varepsilon \frac{y+1}{\alpha^2} + \varepsilon^2 R_1(t, x, y, z, w) \\ w \\ bx - bz + \varepsilon \frac{w+1}{\alpha^2} + \varepsilon^2 R_2(t, x, y, z, w), \end{pmatrix}.$$

In the region $\mathcal{R}_2 = \{x < 0 \text{ and } z > 0\}$ we have

$$\mathcal{X}_2 = \begin{pmatrix} y \\ -ax + z + \varepsilon \frac{y-1}{\alpha^2} + \varepsilon^2 R_1(t, x, y, z, w) \\ w \\ bx - bz + \varepsilon \frac{w+1}{\alpha^2} + \varepsilon^2 R_2(t, x, y, z, w), \end{pmatrix}.$$

In the region $\mathcal{R}_3 = \{x < 0 \text{ and } z < 0\}$ we have

$$\mathcal{X}_3 = \begin{pmatrix} y \\ -ax + z + \varepsilon \frac{y-1}{\alpha^2} + \varepsilon^2 R_1(t, x, y, z, w) \\ w \\ bx - bz + \varepsilon \frac{w-1}{\alpha^2} + \varepsilon^2 R_2(t, x, y, z, w), \end{pmatrix}.$$

Finally, in the region $\mathcal{R}_4 = \{x > 0 \text{ and } z < 0\}$ we have

$$\mathcal{X}_4 = \begin{pmatrix} y \\ -ax + z + \varepsilon \frac{y+1}{\alpha^2} + \varepsilon^2 R_1(t, x, y, z, w) \\ w \\ bx - bz + \varepsilon \frac{w-1}{\alpha^2} + \varepsilon^2 R_2(t, x, y, z, w), \end{pmatrix}.$$

To study the types of the sets \mathcal{M}_{ij} (see Appendix A), we have to compute Lie derivative of the functions g_1 and g_2 with respect to the vector fields \mathcal{X}_i for $i = 1, 2, 3, 4$, i.e.

$$(\mathcal{L}_{\mathcal{X}_i})(g_j)(x, y, z, w) = \langle \nabla g_j, \mathcal{X}_i \rangle(x, y, z, w).$$

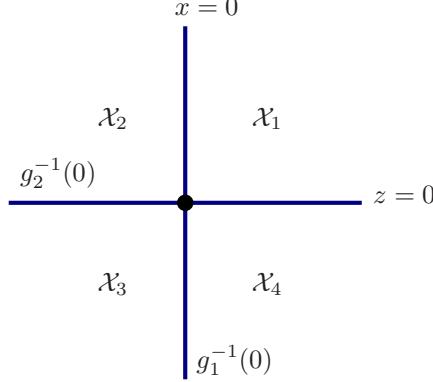


FIGURE 4. Four different vector fields.

Proceeding with these calculations we have

$$\begin{aligned} (\mathcal{L}_{X_1})(g_1)(x, y, z, w) &= (\mathcal{L}_{X_2})(g_1)(x, y, z, w) = -y, \\ (\mathcal{L}_{X_2})(g_2)(x, y, z, w) &= (\mathcal{L}_{X_3})(g_2)(x, y, z, w) = -w, \\ (\mathcal{L}_{X_3})(g_1)(x, y, z, w) &= (\mathcal{L}_{X_4})(g_1)(x, y, z, w) = -y, \\ (\mathcal{L}_{X_1})(g_2)(x, y, z, w) &= (\mathcal{L}_{X_4})(g_2)(x, y, z, w) = -w. \end{aligned}$$

Hence we can conclude that in the set

$$\mathcal{T} = \{(x, y, 0, 0)\} \cup \{(0, 0, z, w)\},$$

the flow is tangent to the discontinuous set, and in any other point the flow cross the set of discontinuity.

Using the coordinates defined in (13), we have that

$$\mathcal{T} = \left\{ \left(X, Y, -\frac{\omega_2}{\omega_1}X, -Y \right) \right\} \cup \left\{ \left(X, Y, \beta \frac{\omega_2}{\omega_1}X, \beta Y \right) \right\}.$$

Observe that the periodic orbits given by Lemma 5 filling the planes $\{(X, Y, 0, 0)\}$ and $\{(0, 0, X, Y)\}$, except the origin, do not reach the set \mathcal{T} . Thus, for $|\varepsilon| > 0$ sufficiently small, there exists a neighborhood of the planes $\{(X, Y, 0, 0)\}$ and $\{(0, 0, X, Y)\}$ such that the orbits cross the set of discontinuity. In other words, the Crossing Hypothesis is satisfied.

Studying the changes of the sign of the function $X_{X_0, Y_0}(\tau)$ for $t \in [0, T_1]$ we conclude that the non-smooth functions (4) and (5) are given by

$$\mathcal{F}_1(X_0, Y_0) = \begin{cases} -\frac{2\sqrt{\Delta}\pi}{\omega_1}Y_0 + \frac{4(a+b+\sqrt{\Delta})}{\omega_1\sqrt{1+\frac{Y_0^2}{X_0^2}}} & \text{if } X_0 > 0, \\ -\frac{2\sqrt{\Delta}\pi}{\omega_1}Y_0 - \frac{4(a+b+\sqrt{\Delta})}{\omega_1\sqrt{1+\frac{Y_0^2}{X_0^2}}} & \text{if } X_0 < 0, \end{cases}$$

$$\mathcal{F}_2(X_0, Y_0) = \begin{cases} \frac{2\sqrt{\Delta}\pi}{\omega_1}Y_0 - \frac{4(a+b+\sqrt{\Delta})}{\omega_1\sqrt{1+\frac{Y_0^2}{X_0^2}}} & \text{if } X_0 > 0, \\ \frac{2\sqrt{\Delta}\pi}{\omega_1}Y_0 + \frac{4(a+b+\sqrt{\Delta})}{\omega_1\sqrt{1+\frac{Y_0^2}{X_0^2}}} & \text{if } X_0 < 0, \end{cases}$$

This system has all solutions inside a periodic orbit of the unperturbed systems passing through

$$(X_0^*, Y_0^*) = \left(\frac{\sqrt{2}}{\sqrt{\Delta}\pi}(a+b+\sqrt{\Delta}), \frac{\sqrt{2}}{\sqrt{\Delta}\pi}(a+b+\sqrt{\Delta}) \right).$$

It is easy to check that this solution are simple. So, by Theorem 1 we have one periodic solution of the non-smooth perturbed double pendulum. This completes the proof of the corollary. \square

Proof of Corollary 4. This proof is completely analogous to the proof of Corollary 2. \square

APPENDIX A: BASIC CONCEPTS ON FILIPPOV SYSTEMS

We say that a vector field $X : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *Piecewise Continuous* if the domain D can be partitioned in a finite collection of connected, open and disjoint sets D_i , $i = 1, \dots, k$, such that, the vector field $X|_{\overline{D}_i}$ is continuous for $i = 1, \dots, k$.

We denote by $S_X \subset \partial D_1 \cup \dots \cup \partial D_k$ the set of points where the vector field X is discontinuous. By assumptions, the set S_X has measure zero.

If $\mathcal{M} \subset S_X$ is a manifold of codimension one, then \mathcal{M} can be decomposed as the union of the closure of the regions (see Figure 5):

$$\begin{aligned} \Sigma^c &= \{x \in \mathcal{M} : (Xh)(Yh)(x) > 0\}; \\ \Sigma^e &= \{x \in \mathcal{M} : (Xh)(x) > 0 \text{ e } (Yh)(x) < 0\}; \\ \Sigma^s &= \{x \in \mathcal{M} : (Xh)(x) < 0 \text{ e } (Yh)(x) > 0\}. \end{aligned}$$

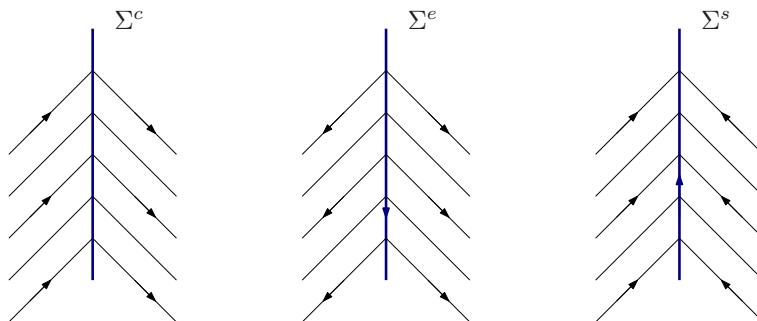


FIGURE 5. Crossing region (Σ^c), escaping region (Σ^e) and sliding region (Σ^s).

For $p \in \Sigma^e \cup \Sigma^s$ we define the *Sliding Vector Field* as

$$(20) \quad Z_s(p) = \frac{1}{(Yh)(p) - (Xh)(p)} ((Yh)(Xh)(p) - (Xh)(Yh)(p)).$$

Consider the following equation

$$(21) \quad \dot{x} = X(x),$$

where $X : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a piecewise continuous vector field. The local solution of the equation (21) passing through a point $p \in \mathcal{M}$ is given by the Filippov convention:

- (i) for $p \in \Sigma^c$ such that $(Xh)(p), (Yh)(p) > 0$ and taking the origin of time at p , the trajectory is defined as $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \in I_p \cap \{t < 0\}$ and $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \in I_p \cap \{t > 0\}$. For the case $(Xh)(p), (Yh)(p) < 0$ the definition is the same reversing time;
- (ii) for $p \in \Sigma^e \cup \Sigma^s$ such that $Z_s(p) \neq 0$, $\varphi_Z(t, p) = \varphi_{Z_s}(t, p)$ for $t \in I_p \subset \mathbb{R}$.

Here φ_W denotes the flow of a vector field W .

For more details about discontinuous differential equation see Filippov's book [4].

APPENDIX B: BASIC RESULTS ON AVERAGING THEORY

We present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T -periodic solutions from differential systems of the form

$$(22) \quad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}) + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x}, \varepsilon),$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $G_0, G_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ and $G_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$ are C^2 functions, T -periodic in the first variable, and Ω is an open subset of \mathbb{R}^n . The main assumption is that the unperturbed system

$$(23) \quad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}),$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the system (23) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$(24) \quad \dot{\mathbf{y}} = D_{\mathbf{x}}G_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}.$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (24), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^k$ the projection of \mathbb{R}^n onto its first k coordinates; i.e. $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

We assume that there exists a k -dimensional submanifold \mathcal{Z} of Ω filled with T -periodic solutions of (23). Then an answer to the problem of bifurcation of T -periodic solutions from the periodic solutions contained in \mathcal{Z} for system (22) is given in the following result.

Theorem 6. *Let V be an open and bounded subset of \mathbb{R}^k , and let $\beta : \text{Cl}(V) \rightarrow \mathbb{R}^{n-k}$ be a C^2 function. We assume that*

- (i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \text{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (23) is T -periodic;
- (ii) for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (24) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n - k)$ zero matrix, and in the lower right corner a $(n - k) \times (n - k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{G} : \text{Cl}(V) \rightarrow \mathbb{R}^k$

$$(25) \quad \mathcal{G}(\alpha) = \xi \left(\frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right).$$

If there exists $a \in V$ with $\mathcal{G}(a) = 0$ and $\det((d\mathcal{G}/d\alpha)(a)) \neq 0$, then there is a T -periodic solution $\varphi(t, \varepsilon)$ of system (22) such that $\varphi(0, \varepsilon) \rightarrow \mathbf{z}_a$ as $\varepsilon \rightarrow 0$.

Theorem 6 goes back to Malkin [6] and Roseau [7], for a shorter proof see [3].

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